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An extension of interior point potential reduction algorithm to solve general LCPs

Laura Di Giacomo

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Abstract Linear complementarity problems (LCP) may be solved if their condition number, defined appropriately is positive. This paper presents a transformation for the LCP which satisfy mild conditions to a form which will have a positive condition number. For these extensive classes an Interior point method can be used to solve it.

Keywords Interior point method · Linear complementarity problem · Condition number · Algorithm · Extension · Condition number

1 Introduction

Much research has been conducted on suitable algorithms to solve the linear complementarity problems (LCP) with Interior Point polynomial algorithms by extending the matrix classes and generalize the types of problems that can be solved by this method [4,5].

Various matrix classes can be distinguished by their condition number with regard to their degree of difficulty to solve the problem when a potential reduction algorithm is used. The condition number derived will of course depend on the data of the problem, that is the coefficient matrix and the affine vector which defines the problem [5].

Thus the solvability of the LCP by an Interior Point method in polynomial time may be recognized, for many matrix classes, by the value of the condition number of the problem.

The aim of this paper is to present an extension of this "condition number based" algorithm originally proposed [5].

The generalization allows the algorithm to be applied to any LCP if its solution set is bounded from above and the coefficient matrix as well as a matrix derived from it satisfy certain conditions. The solution is then obtained by applying an appropriate normalization transformation and solving the resulting problem by a well known polynomial algorithm for LCP problems [3].

L. Di Giacomo (✉)
LIX, Laboratoire d'Informatique, Ècole Polytechnique, Palaiseau, France
e-mail: digiacomo@lix.polytechnique.fr

The outline of the paper is following. After the introduction, in section two certain preliminary results will be presented to make the paper reasonably self contained. In Sect. 3 the extension of the algorithm will be described, while its convergence properties will be analysed in Sect. 4. In Sect. 5 some computational examples will be given, while the appropriate conclusions will terminate the paper.

2 An interior point potential reduction algorithm for general LCPs

Given a linear complementarity problem (LCP(q, M))

$$y = Mx + q \tag{1}$$

$$x, y \geq 0 \tag{2}$$

$$x^T y = 0 \tag{3}$$

where M is an $n \times n$ square matrix and q is a known vector of dimension n.

Consider the primal-dual potential function of a linear complementarity problem defined as follows:

$$\Psi(x, y) = (n + \rho) ln(x^T y) - \sum_{j=1}^{n} ln(x_j y_j) \tag{4}$$

with $\rho > 0$, and a F "feasible region":

$$F = \{(x, y) : y = Mx + q, \quad x \geq 0, \quad y \geq 0\} \tag{5}$$

define the *intF* as the interior of F:

$$intF = \{(x, y) : y = Mx + q, \quad x > 0, \quad y > 0\} \tag{6}$$

To achieve a potential reduction, it is possible to use the scaled gradient projection method, then a linear program subject to an ellipsoid constraint is solved with respect to the variables d_x, d_y where these variables indicate the directions of descent from the previous point to the new point [3].

At the k-th iteration the following linear program is solved, subject to an ellipsoid constraint:

$$\text{Min} \quad Z = \nabla^T \Psi_{x^k} d_x + \nabla^T \Psi_{y^k} d_y \tag{7}$$

$$\text{s.t.} \quad d_y = M d_x \tag{8}$$

$$1 > \alpha^2 \geq ||(X^k)^{-1} d_x||^2 + ||(Y^k)^{-1} d_y||^2 \tag{9}$$

This problem may be solved by the following algorithm:

Algorithm 2.1 Given $x^0, y^0 > 0, \epsilon > 0, y^0 = Mx^0 + q, \quad k := 0$
While $(x^k)^T (y^k) > \epsilon$ Do;

1. Compute

$$\pi = \left(\left(Y^k\right)^2 + M\left(X^k\right)^2 M^T\right)^{-1} \left(Y^k - MX^k\right)\left(X^k y^k - \left(\frac{(x^k)^T (y^k)}{n + \rho}\right) e\right) \tag{10}$$

$$p^k = \begin{pmatrix} p_x^k \\ p_y^k \end{pmatrix} = \begin{pmatrix} \frac{n+\rho}{(x^k)^T (y^k)} X^k (y^k + M^T \pi) - e \\ \frac{n+\rho}{(x^k)^T (y^k)} Y^k (x^k - \pi) - e \end{pmatrix} \tag{11}$$

Select

$$\alpha = \min\left\{\frac{||p^k||}{n+\rho+2}, \frac{1}{n+\rho+2}\right\} \leq \frac{1}{2}; \tag{12}$$

So to find d_x, d_y solve:

$$\begin{pmatrix} (X^k)^{-1}d_x \\ (Y^k)^{-1}d_y \end{pmatrix} = -\alpha \frac{p^k}{||p^k||} \tag{13}$$

2. Let $x^{k+1} = x^k + d_x$, $y^{k+1} = y^k + d_y$;

3. Let $k := k+1$ Return 1. □

The expression for $||p^k||$ as indicated in the algorithm can be considered the potential reduction at the k-th iteration of the objective function. For any x, y let

$$g(x, y) = \frac{n+\rho}{x^T y} Xy - e \tag{14}$$

$$H(x, y) = 2I - (XM^T - Y)(Y^2 + MX^2M^T)^{-1}(MX - Y) \tag{15}$$

which is a positive semi-definite matrix. Thus

$$||p^k||^2 = g^T(x^k, y^k)H(x^k, y^k)g(x^k, y^k) \tag{16}$$

This can also be indicated as $||g(x, y)||_H^2 = g^T(x, y)H(x, y)g(x, y)$.

Define a condition number for the LCP(q, M) as:

$$\gamma(M, q, \epsilon) = inf\{||g(x, y)||_H^2 \mid x^T y \geq \epsilon, \Psi(x, y) \leq \Psi^0, (x, y) \in int(F)\} \tag{17}$$

where the values of the problem for the previous iteration are indicated by a null superscript, as $\Psi^0(x, y)$ or x^0.

The following theorem and corollary can be proved [4]:

Theorem 2.1 *The potential reduction Algorithm 2.1 with $c_1 x \leq \rho \leq c_2 x$ for some constants $0 < c_1 < c_2$ solves the LCP(q, M) for which $\gamma(M, q, \epsilon) > 0$ in $O\left(\frac{\Psi^0 + \rho ln(\frac{1}{\epsilon}) - nln(n)}{\xi(\gamma(M,q,\epsilon))}\right)$ iterations where $\xi(\gamma(M, q, \epsilon)) = Min\{\frac{\gamma(M,q,\epsilon)}{2(n+\rho+2)}, \frac{1}{2(n+\rho+2)}\}$ and each iteration solves a system of linear equations in at most $O(n^3)$ operations.*

Corollary 2.1 *An instance of an LCP(q, M) is solvable in polynomial time if $\gamma(M, q, \epsilon) > 0$ and if $\frac{1}{\gamma(M,q,\epsilon)}$ is bounded above by a polynomial in $ln(\frac{1}{\epsilon})$ and n.* □

The condition number $\gamma(M, q, \epsilon)$ represents the degree of difficulty for the Potential Reduction Algorithm in solving the LCP(q, M). The larger the condition number results, the easier can the problem be solved [4].

Thus the condition number for LCPs provides a criterion to subdivide given instances of LCP(q, M) into classes and those that can be solved in polynomial time may be indicated.

To this end the following definitions are important.

Consider the set:

$$\sum^{+}(M, q) = \left\{\pi \mid x^T y - q^T \pi < 0, x - \pi > 0, y + M^T \pi > 0 \text{ for some } (x, y) \in int(F)\right\} \tag{18}$$

Definition 2.1 let G be a set of LCP(q, M) such that the following conditions are satisfied:

$$G = \left\{ (M, q) \mid \text{int}(F) \neq \emptyset, \quad \sum^{+}(M, q) = \emptyset \right\} \tag{19}$$

Lemma 2.1 *Let $\sum^{+}(M, q)$ be empty for an LCP(q, M). Then for $\rho \geq n + \sqrt{2n}$, $\gamma(M, q, \epsilon) \geq 1$.*

Lemma 2.2 *Let $\{\pi \mid x^T y - q^T \pi > 0, x - \pi > 0, y + M^T \pi > 0$ for some $(x, y) \in int(F)\}$ be empty for an LCP (q, M). Then for $0 < \rho \leq n - \sqrt{(2n)}$, there results $\gamma(M, q, \epsilon) \geq 1$*

Thus directly or by showing that $(M, q) \in G$ or by showing that Lemma 2.2 holds, the instance is solvable in polynomial time with the potential reduction Algorithm 2.1 , by Corollary 2.1.

The following classes can be shown to have one of these properties and therefore solvable in polynomial time by this algorithm, [4,5]:

- Let $\rho > n$ then for a positive semi-definite diagonal matrix M and any $q \in R^n$, $\gamma(M, q, \epsilon) \geq n$
- Let $\rho \geq n + \sqrt{2n}$ then for any positive semidefinite matrix M and any $q \in R^n$, $\gamma(M, q, \epsilon) \geq 1$.
- If M is a copositive matrix and $q \geq 0$ then $(M, q) \in G$.
- If M^{-1} is a copositive matrix and $M^{-1} q \leq 0$ then $(M, q) \in G$.

It is shown that the potential reduction algorithm will solve, under general conditions, the LCP(q, M) when M is a $\mathcal{P}$-matrix and when M is a row-sufficient matrix, [4]:

Theorem 2.2 *Let $\Psi(x^0, y) \leq O(nln(n))$ and M be a $\mathcal{P}$-matrix. Then the potential reduction algorithm terminates at $x^T y < \epsilon$ in $O(n^2 max\{\frac{|\lambda|}{\theta(n)}, 1\} ln(\frac{1}{\epsilon}))$ iterations and each iteration uses at most $O(n^3)$ arithmetic operations.*

The bound indicates that the algorithm is a polynomial-time algorithm if $\frac{|\lambda|}{\theta(n)}$ is bounded above by a polynomial in $ln(\frac{1}{\epsilon})$ and n.

Theorem 2.3 *Let $\rho > 0$ and fixed. For a row-sufficient matrix M and $\{(x, y) \in F \mid \Psi(x, y) \leq \Psi^0\}$ bounded then $\gamma(M, q, \epsilon) > 0$*

Since for the LCP(q, M) defined by this class of matrices the condition number is bounded away from zero, the potential reduction algorithm will solve this class of problems [4].

3 Extension of the algorithm

Consider an LCP(q, M) with a nonsingular coefficient matrix M, for which moreover $(I - M)$ is nonsingular and the solution set of the LCP(q, M) is bounded from above. This LCP can be so indicated:

$$Mu + q - v = 0 \tag{20}$$

$$u, v \geq 0 \tag{21}$$

$$u^T v = 0 \tag{22}$$

where $u, v, q \in R^n$. Suppose that the LCP solution set $S = \{u, v | Mu + q - v = 0, u, v \geq 0, u^T v = 0\}$ is bounded above by a vector $(m_1^T, m_2^T)^T \in R^{2n}$.

Without loss of generality rewrite the problem with variables:

$$u = Dx \tag{23}$$

$$v = Dy \tag{24}$$

by choosing appropriate diagonal positive matrix D, given as:

$$D_{ii} > \max_i\{(2m_1)_i, (2m_2)_i\}, \quad \forall i = 1, 2, \ldots, n \tag{25}$$

Thus:

$$y = D^{-1}v = D^{-1}(Mu + q) = (D^{-1}MD)x + D^{-1}q \tag{26}$$

$$\frac{1}{2}e \geq x, y \geq 0 \tag{27}$$

$$x^T y = 0 \tag{28}$$

This defines an equivalent LCP in bounded variables $0 < x < \frac{1}{2}e$ and $0 < y < \frac{1}{2}e$, which without loss of generality will be indicated as:

$$\hat{M}x + \hat{q} - y = 0 \tag{29}$$

$$x, y \geq 0 \tag{30}$$

$$x^T y = 0 \tag{31}$$

For the potential reduction algorithm to solve general LCPs, it is required that $x > 0$ and $y > 0$.

Lemma 3.1 *For a nonsingular M the matrices $\hat{M} = D^{-1}MD$, $(I - \hat{M})$, $(I - XY\hat{M})$ and $(-Y + \hat{M}X)$ are all nonsingular, as well as: $(I - XY^{-1}\hat{M})$, $(I - XY\hat{M})$ and $(I + XY^{-1}\hat{M})$.*

Proof The nonsingularity of $\hat{M}$ follows immediately from the product with two nonsingular diagonal matrices.

Since $(I - M)$ is nonsingular by assumption, consider:

$$\left(I - \hat{M}\right) = \left(I - D^{-1}MD\right) = D^{-1}\left(DD^{-1} - M\right)D \tag{32}$$

and so $(I - \hat{M})$ has full rank it is the product of three nonsingular matrices, as D is a positive nonsingular diagonal matrix.

Now, given that $0 < X < I$ and $0 < Y < I$ there results as $\lim_{n\to\infty}(XYM)^n = 0$:

$$(I - XY\hat{M})(I + XY\hat{M} + (XY\hat{M})^2 + (XY\hat{M})^3 + \cdots) = I \tag{33}$$

so $(I - XY\hat{M})$ is nonsingular.

Consider now $(-Y + \hat{M}X)$, it is to be proved that this matrix is also nonsingular. Expand the matrix sum and product according to the Sherman-Morrison-Woodbury formula to obtain, [2]:

$$(-Y + \hat{M}X)^{-1} = -Y^{-1} - Y^{-1}\hat{M}(I - XY^{-1}\hat{M})^{-1}XY^{-1} \tag{34}$$

as long as Y and $(I - XY^{-1}\hat{M})$ are nonsingular. The first follows because of the assumed property of the algorithm, the second follows from above.

In an analogous fashion the other matrix products can be demonstrated. □

Corollary 3.1 *Under the conditions of Lemma 3.1* $(Y + \hat{M}X)$ *is nonsingular.* □

The following additional lemma is also required.

Lemma 3.2 *For all LCP*(q, M) *with nonsingular matrices* M *and* $(I - M)$ *transformed to the form given by the system 26–28 so that for any feasible solution* $(x, y) \in int(F)$ $0 < X < I$, $0 < Y < I$, *there results* $g(x, y) = \frac{n+\rho}{x^T y} Xy - e \neq 0$

Proof By construction $e^T Xy = x^T y > 0$. For any $g(x, y)$, $\frac{n+\rho}{x^T y} Xy > 0$ by the properties of a feasible solution of the interior point method considered.

Assume $g(x, y) = 0$ it is shown that this leads to a contradiction.

If $g(x, y) = 0$ $\frac{n+\rho}{x^T y} X_i y_i - 1 = 0$ for any $i = 1, 2, \ldots, n$. As $\frac{n+\rho}{x^T y} Xy > 0$ then

$$\sum_{i=1}^{n} \left(\frac{n+\rho}{x^T y} X_i y_i - 1 \right) = 0 \tag{35}$$

and therefore $\frac{n+\rho}{x^T y} e^T Xy - e^T e = 0$. This implies that $n + \rho - n = 0$ so $\rho = 0$, but this contradicts the thesis. Therefore $g(x, y) \neq 0$ □

4 Convergence results

Theorem 4.1 *For all LCP*(q, M) *with* $\rho > 0$ *and nonsingular matrices* M *and* $(I - M)$ *transformed to the form given by the system 26–28 so that for any feasible solution* $(x, y) \in int(F)$, *the condition number for the LCP is given by* $\gamma(\hat{M}, \hat{q}, \epsilon) > 0$ *for some* $\rho > 0$

Proof Assume that $||g(x, y)||_H^2 = 0$ and expand it in terms of its factors as given by Eq. 15.

$$||g(x, y)^T [2I - (X\hat{M}^T - Y)(Y^2 + \hat{M}X^2\hat{M}^T)^{-1}(\hat{M}X - Y)]g(x, y)|| = 0 \tag{36}$$

As $g(x, y) \neq 0$ by Lemma 3.2 then:

$$g(x, y)^T [2I - (X\hat{M}^T - Y)(Y^2 + \hat{M}X^2\hat{M}^T)^{-1}(\hat{M}X - Y)] = 0 \tag{37}$$

Thus:

$$0 = g(x, y)^T [2(\hat{M}X - Y)^{-1} - (X\hat{M}^T - Y)(Y^2 + \hat{M}X^2\hat{M}^T)^{-1}] \tag{38}$$

$$0 = g(x, y)^T [2(\hat{M}X - Y)^{-1}(Y^2 + \hat{M}X^2\hat{M}^T) - (X\hat{M}^T - Y)] \tag{39}$$

$$0 = g(x, y)^T [2(Y^2 + \hat{M}X^2\hat{M}^T) - (\hat{M}X - Y)(X\hat{M}^T - Y)] \tag{40}$$

which results:

$$0 = g(x, y)^T (X\hat{M}^T - Y)^T (X\hat{M}^T - Y) \tag{41}$$

since X and Y are diagonal matrices, and $(X\hat{M}^T + Y)$ is nonsingular.

The contradiction establishes the theorem. □

Remark 4.1 The positive semidefinite matrix $H(x, y)$ may result null. Under the assumptions proposed in the extension, this cannot happen by Corollary 3.1.

Theorem 4.1 permits to extends the results given above in Theorem 2.1 to the following theorem and to the subsequent corollary.

Theorem 4.2 *The potential reduction Algorithm 2.1 with $c_1 x \leq \rho \leq c_2 x$ for some constants $0 < c_1 < c_2$ solves an LCP(q, M) with nonsingular coefficient matrix M and nonsingular matrix $(I - M)$ in $O\left(\frac{\Psi^0 + \rho ln(\frac{1}{\epsilon}) - nln(n)}{\xi(\gamma(\hat{M}, \hat{q}, \epsilon))}\right)$ iterations where $\xi(\gamma(\hat{M}, \hat{q}, \epsilon)) = Min\{\frac{\gamma(\hat{M}, \hat{q}, \epsilon)}{2(n+\rho+2)}, \frac{1}{2(n+\rho+2)}\}$ and each iteration solves a system of linear equations in at most $O(n^3)$ operations.* □

Proof By Theorem 2.1, 4.1.

Corollary 4.1 *An instance of an LCP(q, M) with a nonsingular coefficient matrix M and nonsingular matrix $(I - M)$ is solvable in polynomial time if $\frac{1}{\gamma(\hat{M}, \hat{q}, \epsilon)}$ is bounded above by a polynomial in $ln(\frac{1}{\epsilon})$ and n.* □

This therefore partially generates the important results formulated previously [4,5].

5 Examples

The aim of this section is to indicate the form of problems that can be solved by the extension of the algorithm, but which cannot be solved by the original proposed algorithm [4].

Of course the matrices of the problems considered do not belong to the classes which can be solved by the original algorithm, but are easy problems to solve by other methods.

Five LCP will be defined, four of which cannot be solved by the original algorithm, but which can be solved by the extension proposed and then some computational results on 20 similar types of problems, defined by similar matrices are proposed to provide the reader with some significance.

e.g. A Consider the following problem:

$$Lcp(M, q) = \left\{ \begin{pmatrix} -1.0 & 2.0 & 2.0 \\ 2.0 & -3.0 & -1.0 \\ -1.0 & 2.0 & 3.0 \end{pmatrix}, \begin{pmatrix} -2.0 \\ 5.0 \\ -3.0 \end{pmatrix} \right\} \tag{42}$$

The Pardalos-Ye algorithm yields a solution which is not complementary, but it indicates at termination a condition number of approximately 35 as in the second algorithm.

In the proposed algorithm the solution give is $x = (0.0, 0.0, 1.0)$ which is a correct solution. The solution times of both problems are not significant due to size.

e.g. B Consider the following problem:

$$Lcp(M, q) = \left\{ \begin{pmatrix} -1.0 & 2.0 & -4.0 \\ 2.0 & -3.0 & 0.0 \\ -4.0 & 0.0 & 3.0 \end{pmatrix}, \begin{pmatrix} -2.0 \\ 5.0 \\ 3.0 \end{pmatrix} \right\} \tag{43}$$

Again, the former algorithm yield a complementarity value of 2.1 and a condition number of 31. In the proposed algorithm a solution is determined $x = (0.0, \frac{5}{3}, 0.0)$ and a condition number of 11.1 is indicated at convergence.

e.g. C Consider the following problem:

$$Lcp(M, q) = \left\{ \begin{pmatrix} -1.0 & 2.0 & 2.0 \\ 2.0 & -3.0 & -1.0 \\ -1.0 & 2.0 & -3.0 \end{pmatrix}, \begin{pmatrix} -2.0 \\ 5.0 \\ -3.0 \end{pmatrix} \right\} \tag{44}$$

Table 1 Computational results on linear complementarity problems of different algorithms

	Solved	Not solved	Total
Lcp-goal	20	...	20
Pardalos-Ye	13	7	20
Di Giacomo	17	3	20

The original algorithm does not yield a solution, but indicates a complementarity value of 0.83 and a condition number at the last iteration 496.6, while the proposed algorithm yields a solution $x = (0.0, \frac{5}{3}, 0.0)$ with a asymptotic condition number of 10.7.

e.g. D Consider the following problem:

$$Lcp(M, q) = \left\{ \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}, \begin{pmatrix} -1.0 \\ -1.0 \\ 0.5 \\ 0.5 \\ -2.0 \end{pmatrix} \right\} \tag{45}$$

The original algorithm defaulted with a complementarity value of 0.196 and a terminal condition number of 117.3. Instead the proposed algorithm determined an optimal solution $x = (1.0, 1.0, 0.0, 0.0, 0.0)$ with a condition number of 147.2.

e.g. E Consider the following problem:

$$Lcp(M, q) = \left\{ \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 1.0 & 1.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}, \begin{pmatrix} 1.0 \\ 1.0 \\ -0.5 \\ -0.5 \\ -2.0 \end{pmatrix} \right\} \tag{46}$$

Here both algorithms do not find a complementarity solution, since this problem has no complementarity solution. The respective complementarity values identified for the original and the proposed algorithms are 1.91 and 0.447 and the condition numbers reported were respectively 98.6 and 114.8.

A number of similar problems were also tried and the results were compared to the algorithm LCPGOAL, proposed [1] and are presented in Table 1. The elaborations effected indicate that there is more work to be done in the analysis of the role of the condition number and how it changes with respect to the given problems during the iterations, in line with the analysis proposed [5]. This however is a problem to be studied in a future paper.

6 Conclusions

In this paper an extension of an algorithm to solve the LCP problems is presented based on the 'condition number' of the given coefficient matrix, by considering a transformation so as to solve the LCP for more general classes of matrices.

The results allow this extended algorithm to be applied to all LCP(q, M) which have a nonsingular coefficient matrix and for which, moreover $(I - M)$ is nonsingular, and its solution set is bounded from above, by carrying out an appropriate normalization transformation.

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